

Mathematics Teaching in Ontario's Public Schools

June 27, 2016

Philip. A. Sullivan, Ph. D., P.Eng.

Professor Emeritus, Institute for Aerospace Studies, University of Toronto

(Email: psullivan@utias.utoronto.ca)

God created the natural numbers; everything else is man's handiwork.

— German mathematician Leopold Kronecker [1823-1891]

Civilization advances by extending the number of important operations which we can perform without thinking about them.

— English philosopher and mathematician Alfred North Whitehead [1861-1947]

The principal and head of the mathematics department [at my son's school] told parents to their faces that long division was obsolete, that the school would never teach it, and that parents who thought the school should teach it were dinosaurs.

— Johns Hopkins University professor of mathematics W. S. Wilson

Abstract

This article reviews literature from academic and other sources on problems in mathematics education for North American school grades K-12. It is shown that, especially for the critical grades K-6, the dominant educators' ideas are controversial, being subject to strong criticism by other educators, by mathematicians, by cognitive psychologists, and by other concerned professionals. This extends to characterization of the subject matter of mathematics as well as to both curriculum and teaching practices. Aspects of the Ministry of Ontario's curriculum and advisory literature are discussed. The author concludes that, in view of Ontario school childrens' poor showing on international tests, detailed reviews of curriculum structure, of textbooks, of teaching practices and of evaluation methods are all needed. Two recent critiques of U.S. mathematics education are used to suggest a path for reform.¹

Introduction

That something is seriously wrong with mathematics teaching in Ontario's public schools can no longer be denied: the province's colleges are being forced to "divert millions of dollars from their operating budgets to support at-risk students through a multitude of remediation interventions." (Dziwak, 2014). In a C. D. Howe Institute commentary, University of Winnipeg mathematician Anna Stokke (2015) shows that the problem exists across Canada. Furthermore, it is widely recognised by Canadian university administrators: in *University Affairs*, Anne Kershaw (2010) observes "The math skills of students entering Canadian universities have declined sharply in recent years, with many students unable to do basic arithmetic." It is endemic to North America and to other Western countries. Consequently, according to Johns Hopkins' W. S. Wilson, in the USA "73 percent of the mathematics taught in colleges ... is really K-12 mathematics." I argue that Ontario's K-12 mathematics curriculum and teaching policies both need reform. I suggest that, in attempting to correct chronic problems originating in the notorious "New Math" curriculum of the 1960's, beginning in the 1990's educators have themselves developed counterproductive policies.

Not usually being mathematicians, mathematics educators have inhibited or even opposed the necessary reforms in several ways. Four stand out: (1), misinterpretation of the implications of

¹ The diagram on page 14 is in colour.

developments in cognitive psychology applicable to mathematics teaching; (2), ignoring or disparaging evidence on the relative merits of minimally guided instruction on the one hand, and closely guided instruction on the other; (3), misrepresentation of the nature of mathematics itself in curriculum design and in text-books; and (4), ill-informed training of teachers of the critical grades K-6. I argue for reform involving: (a), collaboration between mathematicians and mathematics educators in both curriculum design and teacher training; and (b), a much greater emphasis on closely guided instruction along the lines suggested by Stokke (2015) and others. Such an approach might well include extensive use of direct instruction involving worked examples, together with modern technological aids including—when appropriate—calculators. Minimal guidance methods, known variously as *discovery*, *inquiry* and *problem-based* learning can have a role to play, becoming increasingly useful as the student advances.

The erosion in basic skills of North-American school children is widely attributed to advocacy in the early 1990's by the U.S. National Council of Teachers of Mathematics (NCTM). Anne Kershaw's (2010) discussion of the Canadian problem notes that this group promoted a "philosophical shift in math pedagogy to emphasising conceptual thinking from what some saw as rote memorization." Hence I review U.S. developments, providing thereby insights into Ontario's problem. Following comments on the implications of cognitive psychology for mathematics education I discuss recent evidence on the attributes of effective teaching. To make clear the nature of educators' misrepresentations of mathematics I describe the foundations of arithmetic and comment briefly on its history, drawing implications for teaching. I give examples of misrepresentation, including indicators of outright incompetence. I show that Ontario's policies bear the hallmarks of the U.S. state of affairs, thus needing reform. I conclude by using two U.S. critiques as a basis for suggestions about reform.

Comments on K-12 mathematics education in the USA

Educators' ideas about effective teaching are—at best—controversial. In his late 20th Century analysis of U.S. educators' doctrines and practices the University of Virginia's E. D. Hirsch quotes a 1953 critique:

One of the most shocking facts about the field of education is the almost complete absence of rigorous criticism from within. The paean of praise that greets every novel proposal, the closing of ranks that occurs whenever a word of criticism is spoken from the outside and ... the extreme unwillingness of professional educationists to submit their proposals to free public discussion and honest criticism frequently assumes the even uglier form of showering critics, no matter how upright and well-informed, with vituperation and personal abuse. (from *Educational Wastelands*, by historian A. E. Bestor in 1953, in Hirsch, 1999 loc 1461)

Hirsch—himself a professor of education—notes that

It is sad to think that there is nothing outdated in his 1953 description of the educational community's resistance to internal or external disagreement. (Hirsch, 1999, loc 1453)

Is such a savage critique justified, especially as it applies to mathematics education? According to mathematician David Klein's account of the history of 20th Century K-12 maths education it is:

Throughout the 20th century the “professional students of education” have militated for child centered discovery learning and against systematic practice and teacher directed instruction...[b]ecause of the hierarchical nature of mathematics and its heavy dependence at any level on prerequisites, high school and even college mathematics courses have at times been strongly affected by [such] ideas, especially at the end of the 20th Century....[In the 1990’s textbooks and curricula] failed to develop fundamental arithmetic and algebra skills. Elementary school programs encouraged students to invent their own algorithms, while discouraging use of the superior standard algorithms... Student discovery group work was the preferred mode of learning... and the guidelines for discovery projects were at best inefficient and often aimless. [In textbooks] topics from statistics were redundant from one grade level to the next... Mathematical definitions and proofs for the higher grades were generally deficient, missing entirely, or even incorrect. Written and published criticism from many sources, including mathematicians, was widespread in the 1990's and reinforced the convictions of dissatisfied parents. (Klein, 2003)

In the media these disputes are known as “the math wars.” Klein notes that journalists habitually portray them as an extended disagreement between university mathematicians and concerned parents wanting basic skills on the one hand, and educators favouring conceptual understanding and so-called “higher order thinking” on the other. But for the mathematicians—some of whom are very distinguished—nothing could be further from the truth. Furthermore, “by contrast, many of the educational professionals who spoke of ‘conceptual understanding’ lack even a rudimentary knowledge of mathematics.” (Klein, 2003).

This bogus dichotomy is beautifully illustrated by University of California mathematician H-H Wu’s (1999) description of the use of basic manipulations leading to both development of the efficient standard algorithms and a deep understanding of arithmetic’s structure. In this respect, as Wu puts it, “conceptual advances are invariably built on the bedrock of technique.” He drew an apt comparison with musical skills: “A violinist who still worries about fingering positions cannot hope to impress with the beauty of tone or the elegance of phrasing.”

More generally it can be said that many of educators’ pronouncements have attracted widespread criticisms as being both ideologically driven and not consistent with the findings of available evidence. For example, consider the critique by the University of Oregon’s Douglas Carnine.² As a professor of education he nevertheless likens the bulk of his profession to the pre-scientific status of early 20th Century medicine:

[T]he judgements of experts frequently appear to be unconstrained by objective research... Until education becomes the kind of profession that reveres evidence, we should not be surprised to find its experts dispensing unproven methods, endlessly flitting from one fad to another. (Carnine, 2000)

This criticism applies to all disciplines. For example, consider Southern Illinois University School of Medicine’s Jerry Colliver comments on educators’ advocacy of problem-based-learning (PBL):

The review of the literature revealed no convincing evidence that PBL improves knowledge base and clinical performance... The author concludes that the ties between educational theory and research (both basic and applied) are loose at best. (Colliver, 2000)

²Carnine is Director of the University of Oregon’s *National Center to Improve the Tools of Educators*, and is a board member of the non-profit *National Institute for Direct Instruction*.

Former English teacher Daisy Christodoulou's (2014) detailed critique dismisses much of British educators' ideas as "myths." In a foreword to her book, E. D. Hirsch observes that Christodoulou's analysis applies to North American educators, arguing that the "myths [she] identifies are anti-intellectual and (unwittingly) anti-egalitarian." (loc 85) It is therefore not surprising that I and many academics in cognate disciplines are skeptical of much of educational research.

Implications of modern cognitive psychology

Modern cognitive psychology recognizes the crucial distinction to be made between working or conscious memory on the one hand and long-term memory on the other, with learning being essentially a change in long-term memory. Also "almost all information stored in working memory and not rehearsed is lost within 30 seconds ... and the capacity of the working memory is limited to only a very small number of elements... That number may be as low as four, plus or minus one." (Kirschner et al 2006) Furthermore, high skill levels appear to be very domain-specific, being based on extensive knowledge of a particular discipline that is stored in the long term memory. This cognitive structure has basic implications for education suggesting— among other things—that, in the initial stages of learning a discipline, considerable guidance may be required, with the need for such guidance receding as knowledge is acquired.

Yet, apparently ignoring these findings, educators have for decades advocated minimal guidance instruction in various guises such as *discovery learning*, *problem-based learning* and—more recently—using a term borrowed from psychology, *constructivist learning*. According to Hirsch (1999, loc 5196), it appears that educators use the term *constructivism*³

to sanction the practice of "self-paced learning" and "discovery learning." The term implies that only constructed knowledge—knowledge which one finds out for one's self—is truly integrated and understood. It is certainly true that such knowledge is very likely to be remembered and understood, but it is not the case, as constructivists imply, that *only* such self-discovered knowledge will be reliably understood and remembered. This incorrect claim plays on an ambiguity between the technical and nontechnical use of the term 'construct' in the psychological literature. (italics in original)

A paper coauthored by the cognitive science pioneer Nobel Laureate H. A. Simon discusses the implications of cognitive psychology for mathematics education:

There is a frequent misperception that the move [in psychology] from behaviorism to cognitivism⁴ implied an abandonment of the possibilities of decomposing knowledge into its elements for purposes of study and decontextualizing these elements for purposes of instruction... This false rejection of decomposition and decontextualization runs deep in modern mathematics education ... in the draft of the NCTM assessment standard for school mathematics, we find condemnation of the 'essentialist view of mathematical knowledge' which assumes 'mathematics consists of an accumulation of mathematical concepts and skills.'...[We] find frightening the prospect of mathematics education based on such a misconceived rejection of componential analysis. (Anderson et al, 2000)

³ "For a long time it has been known that most memories are not just mechanical recollections but are constructs built on a whole body of relevant experiences." (Hirsch 1999, loc 5203)

⁴ Cognitivism [attempts] to understand mental processes such as perception, thinking, remembering and learning; in contrast, behaviourism... focusses on observable behaviour. (Audi, 1999)

Perhaps the most radical of the constructivists's claims is "denial of the possibility of objective evaluation"(Anderson et al, 2000). The more moderate constructivists advocate focussing on evaluation of the process of learning—in what are considered "authentic" tasks—rather than on the product. They also advocate using multiple perspectives in the evaluation. This has manifested itself in schemes such as the U.S. Government-mandated *National Assessment of Educational Progress* (NAEP). Typically one of its documents states

[A] student will be given 50% (2 points) for the right answer if the justification for the answer 'is not understandable' but will be given 100% (4 points) for the wrong answer if "it does not reflect misunderstanding of either the problem or how to implement the strategy, but rather **seems** to be a copying error or computational error (Anderson et al, 2000, bold in original)

Anderson and his colleagues suggest that "Any time the word 'seems' appears in an assessment, it should be a red flag that the assessors do not know what they are looking for."

U.S. educators' reaction to Anderson et al's critique is a textbook example of their hostility to outside criticism. Among other things, the journal *Educational Researcher* crudely censored it by publishing only that part of the article not explicitly critical of educators' take on constructivism. As Klein (2003) put it, "this decision would have been unremarkable except that the three authors of the article happened to be among the most distinguished cognitive scientists in the world."

On the characteristics of effective teaching

In relation to instructional techniques Anderson et al (2000) draw the following conclusions:

- (1) When students cannot construct knowledge for themselves, they need some instruction. There is very little positive evidence for discovery learning and it is often inferior.... it may be costly in time, and when the search is lengthy or unsuccessful, motivation commonly flags.
- (2) People are sometimes better at remembering information that they create for themselves than information they receive passively, but in other cases they remember as well or better information that is provided than information they create.
- (3) Real competence only comes with extensive practice. The instructional task is not to "kill" motivation by demanding drill, but to find tasks that provide practice while at the same time sustaining interest.

One of the most powerful recent analyses of the characteristics of effective teaching is given by University of Melbourne education professor John Hattie's meta-analysis of the results of over 52,000 studies involving millions of students. His analysis, published in 2009, uses the concept of *effect size*. An effect size $d = 1$ means that, "on average, students receiving [a particular] treatment would exceed 84% of students not receiving that treatment." In the literature, $d = 1$ corresponds to a "large and blatantly obvious" effect (Atherton, 2013). Hattie tested for a total of 28 factors, the dominant five of which are listed in order below:

- #1, ($d = 1.13$): **Feedback** from the student to the teacher about their knowledge, understanding, errors, misconceptions and lack of engagement.
- #2, ($d = 1.04$): **Students' prior cognitive ability**, which is largely beyond the control of the teacher.
- #3, ($d = 1.00$): **Instructional quality**. Excellent teachers can: (1), "identify essential representations of their subject"; (2), "guide learning through classroom interactions"; (3), "monitor learning and

provide feedback”; (4), “attend to affective attributes.” and (5), “influence student outcomes.”
 #4, ($d = 0.82$): **Direct instruction.** “[T]he teacher decides the learning intentions and success criteria, makes them transparent to the students, demonstrates them by modelling, evaluates [the students’] understanding, and re-telling them what they have told by tying it all together with closure.” (Hattie in Christodoulou, 2014, loc 1061)

#5, ($d = 0.65$) **Remediation/feedback** [by the teacher.] (bold in original; see Atherton, 2013)

The other 23 influences range from $d = 0.61$ for “Students disposition to learn” down to $d = 0.12$ for “Finances/money” and $d = -0.05$ for physical attributes such as class size.

The results of Hattie’s and other studies conflict with much of educators’ dogma. Christodoulou quotes Hattie as observing that when he tells teacher trainees about this, they are shocked:

Every year I present lectures to teacher education students and find that they are already indoctrinated with the mantra “constructivism good, direct instruction bad.” When I show them the results of these meta-analyses, they are stunned, and they often become angry at having been given an agreed set of truths and commandments against direct instruction. (Hattie, in Christodoulou 2014, loc 1061)

Thus while excellent teachers use a variety of skills and techniques to advance their students’ learning, Hattie’s analysis underscores the value of direct instruction. In this respect Burbash (2012) describes the results of 14 investigations and meta-analyses undertaken since 1988; all generally favour direct instruction.

On the nature and teaching of mathematics

Leopold Kronecker’s characterization of mathematics quoted at the head of this essay captures a fundamental truth:

While the Greeks chose the geometrical concepts of point and line as the basis of their mathematics, it has become the modern guiding principle that all mathematical statements should be reducible ultimately to statements about the *natural numbers*, 1, 2, 3, ... [Kronecker] pointed out the safe ground on which the structure of mathematics can be built. (Courant and Robbins, 1981, p. 1)

By this we understand that everything is built on the laws established for manipulating the natural numbers, or *positive integers*. Using the symbols a , b and c to represent three different such integers, there are five fundamental laws for addition and multiplication:

Commutative laws (1 and 2) : $a + b = b + a$, and $a \times b = b \times a$,
Associative laws (3 and 4): $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$,
Distributive law (5): $a \times (b + c) = a \times b + a \times c$.

In these laws, brackets indicate operations that are performed first. They correspond to the counting of sets of like objects contained in boxes, with the number zero (0) corresponding to an empty box. *Subtraction* and *division* reverse addition and multiplication, but they do not commute: whereas $3 + 5 = 5 + 3 = 8$, the subtraction $5 - 3$ does not equal $3 - 5$, and $6 \div 3$ does not equal $3 \div 6$.

The ideas of negative and rational numbers build on this foundation. Manipulations of negative numbers are governed by the *rule of signs*: the product $(-1) \times (-1)$ is *defined* as $(+1)$. Manipulations of rational numbers, *the ratio of two integers*, are governed by four *definitions*; these are:

with $\frac{a}{b}$ denoting $a \div b$, and ab denoting $a \times b$,

$$(i), \frac{a}{b} + \frac{c}{d} = \frac{(ad + bc)}{(bd)}; (ii), \left(\frac{a}{b}\right) \times \left(\frac{c}{d}\right) = \frac{(ac)}{(bd)}; (iii), \frac{a}{a} = 1; (iv), \frac{a}{b} = \frac{c}{d} \text{ if } (ad) = (bc).$$

As Courant and Robbins (1981, p. 52) note, whereas the natural numbers are “abstractions from the process of counting finite collections of objects,” the developments of the ideas of negative and rational numbers were prompted by the need to measure quantities such as length, mass and time leading, ultimately, to the concept of the *number line*. These five definitions

are forced upon us if we wish to use the rational numbers as measures of length, areas, etc. ... But they were also chosen to obey the rules of natural number arithmetic... What *can*—and must—be proved is only that on the basis of these definitions the commutative, associative and distributive laws of [natural number] arithmetic are preserved. (Courant and Robbins, 1981, pp. 54-55).

Two implications for teaching mathematics come to mind. First, although educated moderns take for granted the abstractions involved in using the above rules, they are the result of a protracted historical process extending over millenia, with their final evolution to the form described above being achieved relatively late—in the mid-19th Century. For example, the crucial concept of the number line is attributed to the English mathematician John Wallis (1616-1703) as “giving some meaning to negative numbers.” Yet, even in the mid-18th Century some mathematicians considered negative number solutions to equations to be dubious (Nrich Maths, 2016). One of the most prolific mathematicians of all time, Leonhard Euler (1707-1783)—who is revered for his contributions to the calculus currently used by scientists and engineers—gave a thoroughly unsatisfactory rationalization of the rule of signs for negative numbers. In commenting on Euler’s failed attempt, Courant and Robbins (1982, p. 55) note that

It took a long time for mathematicians to realize that the “rule of signs” together with all the other definitions governing negative integers and fractions cannot be “proved.” They are *created* by us in order to obtain freedom of operation while preserving the fundamental laws of arithmetic.

This slow development suggests that it is unrealistic to claim that mathematics can be taught efficiently to children in the critical grades K-6 by anything other than an approach involving closely guided instruction.

The second implication is that, because everything in mathematics builds on the rules for arithmetic, facility in mathematics at all levels requires attaining automaticity in the use of these rules, together with the decimal representation of fractions and with the use of the standard algorithms for the four basic operations on multidigit numbers. If this not attained in the primary grades, learning constructions such as algebra, geometry and beyond—all being grounded in these rules—will be seriously impeded. Philosopher-mathematician Alfred North Whitehead’s observation about the advancement of civilization seems especially apt.

The need to attain automaticity in arithmetic is confirmed by two recent investigations: a large-scale longitudinal study of the development of students’ mathematical abilities in both the USA and Britain, and an investigation which used functional magnetic resonance imaging (fMRI) to study

brain activation involved in simple calculations. The first revealed that

[E]lementary school students' knowledge of fractions and of division uniquely predicts those students' knowledge of algebra and overall mathematics achievement in high school, 5 or 6 years later, even after statistically controlling for other types of mathematical knowledge, general intellectual ability, working memory and family income and education. (Siegler et al., 2012)

The second, conducted at Ontario's Western University, used 19 male and 14 female grade 12 students who had previously consistently demonstrated a range of mathematical abilities, and had taken the U.S.-based Preliminary Scholastic Aptitude Test (PSAT), which is usually taken at Grade 10 or 11. They were asked simple addition/subtraction questions while subject to fMRI. The images showed that those scoring high on the math part of the PSAT tended to use that part of the brain "known to be engaged during arithmetic fact retrieval," while the low scoring students tended to use a region "established to be involved in numerical quantity processing." The authors conclude that these results highlight "the fundamental role that mental arithmetic fluency plays in the acquisition of higher-level mathematical competence." (Price et al., 2013).

Risking stating the obvious, I do not suggest that teaching the rules of arithmetic can be based on explicitly stating them at the outset, nor do I suggest how this task might be achieved. The literature is replete with informed discussions of the challenges involved. For example, psychologist D. T. Willingham (2009) gives useful insights into children's developing cognitive abilities, and suggests ways of achieving an appropriate balance of factual, procedural and conceptual understanding in the early grades. Assuming international math standards to be a suitable guide, the goal should be to achieve full understanding of arithmetic by the end of Grade 6. (Schmidt, 2008)

Mathematics educators' perceptions of mathematics

Professor Wu gives examples supporting his claim that mathematics educators' understanding of the discipline is largely alien to that of mathematicians. He identifies certain specific tendencies:

- (i), *proof-abuse*;
- (ii), *fuzzification of mathematics*;
- (iii), *slighting of basic mathematical techniques*;
- (iv), *obsession with relevance and "real world applications."* (Wu, 1996, italics in original).

I cite examples; the first three being of tendency (i). One U.S. Grade 9 text "makes students verify, by the use of calculators, that for a few choices of the integers M and N , $\sqrt{M \times N} = \sqrt{M} \times \sqrt{N}$. Then, without missing a beat, it asserts that the identity holds in general." Yet this result follows immediately from the definition of the square root and the above manipulation rules. An introductory calculus text discusses the infinite series representation of the function $f(x) = 1/(1 - x)$, which is $1 + x + x^2 + x^3 + x^4 + \dots$, and justifies it by using a computer to print out the first 50 terms. But this series can be easily derived by polynomial long division. In trigonometry an NCTM journal article advocates using a graphics calculator to demonstrate the identity $\sin 2x = 2 \sin x \cos x$, thus having the "supposedly beneficial effect of letting the students avoid 'the rote method of pencil and paper (sic!).'" As Wu (1996) observes, if the article's author had advocated using the graphics calculator as an adjunct to bolster confidence in the formal proof, it should be applauded as "making skillful use of technology in the classroom."

Tendency (ii), *fuzzification*, moves mathematics back into everyday life with all its ambiguity by, among other things, giving incomplete information in the formulation of problems, including introducing undefined quantities. I give below examples in Ontario's Ministry of Education literature. With precision being characteristic of mathematics, this has the "pernicious cumulative influence" of distorting both students' and teachers' perceptions of the subject. (Wu, 1996)

An example of tendency (iii), *technique slighting*, is the multiplication $(ax + b) \times (cx + d)$ in algebra. One Grade 9 text advocates using both the FOIL⁵ mnemonic and a geometric method by cutting a rectangle into four pieces, but not the basic distributive property, as defined—in the first instance—for integers. This is in spite of the text's previous discussion of expanding $a \times (x + b)$ using this basic property, with the students being told that the method is "powerful." (sic!)

On tendency (iv), *obsession with relevance*, illustrating the value of mathematics using applications is an important component of teaching. But Wu objects to the discussion of applications in a way that makes them object of the lesson, arguing that such approaches chronically lack closure by failing to emphasise the properties of the mathematics itself. He cites a glaring example from a high school text: finding the roots of the cubic equation $5x^3 - 12x^2 - 16x + 8 = 0$. The students are asked to find a root numerically, including writing a computer code. But that is all! The properties of cubic equations, such as the basic fact that they have three roots, are not discussed. Should not students learn that the other two can be extracted by long division followed by the standard formula for the solution of quadratic equations?

The prominent activist Ruth Parker⁶ provides a classic example of the obfuscating tendencies of mathematics educators. She "rejects long division and multiplication tables as nonsensical leftovers from a pre-calculator age." In a six-month period she spoke "before 20,000 people ...at the behest of school districts... [urging] audiences to 'let kids play with numbers' [claiming that] they will figure outmost any math concept." (Klein, 2003). Some educators' opposition to the standard computational algorithms seems absurd. In 1994 a mathematics consultant for the Connecticut Department of Education promulgated the false dichotomy between skills and understanding thus:

It's time to recognise that, for many students, real mathematical power on the one hand and facility with multidigit, pencil-and-paper computational algorithms on the other, are mutually exclusive. In fact it's time to acknowledge that continuing to teach these skills to our students is not only unnecessary, but counterproductive and downright dangerous. (Sic!) (Leinwand, 1994)

As the third quote at the head of this essay suggests, educators have vociferously attacked long division, dismissing it as rendered obsolete by the availability of the calculator. While trigonometric tables, logarithmic tables, and square-root extraction techniques—which were universal items in high-school curricula up to the 1970's—are now obsolete, long division remains an essential tool. Mathematicians David Klein and James Milgram (accessed 2016) provide an elegant justification of its continuing role, demonstrating yet again the mathematical ignorance of educators. For example, students' first exposure to long division shows why rational numbers have a repeating decimal representation, but there is much more. They show how long division can be taught so that underlying concepts can be understood, and demonstrate how these concepts apply in more advanced

⁵First, Inner, Outer, Last. This mnemonic promotes the "rote learning" that educators deplore.

⁶CEO of the Mathematics Education Collaborative (www.mec-math.org/about-mec/ruth-parker)

mathematics, such as polynomial long division.

U.S. educators' ill-informed views are propagated by the former dean of education at the University of New Brunswick, Dr. Marian Small; she wields considerable influence in Ontario. Michael Zwaagstra describes revealing exchanges between Dr. Small and concerned parents in Winnipeg, Manitoba; in a presentation Dr. Small

emphasised that there is more than one way to get the correct answer, and encouraged teachers to assign more open-ended and ambiguous math questions to their students, [arguing that] all students would be more likely to get a correct answer... [and] that all ways of solving math problems were equally valid... I asked Dr. Small whether ... teachers should include the standard algorithms as a component of math instruction. She answered that she did not. When I asked her how she reconciled this with her earlier statements that all ways of solving math questions are equally valid, she insisted that the new math techniques were still better... Several math professors in the audience challenged some of Dr. Small's claims about math instruction. At this point Dr. Small shut down questions...she refused to seriously dialogue with anyone who expressed an opposing view. (Zwaagstra, 2012)⁷

Commenting on a series of textbooks Dr. Small has written under the general title *Math Focus*, Zwaagstra observes that they reflect “her random abstract approach ... the standard algorithms ... are almost entirely absent. In their place we find convoluted word problems, confusing instructions and complicated diagrams.”

I examined Dr. Small's text *Uncomplicating Fractions....* (Small, 2014), and concur with Zwaagstra's assessment. Intended as a guide for teachers in the Grades K-7, she asserts

the way we uncomplicate ... is not to come up with a formulaic approach to instruction but to provide the opportunity for a deep and rich understanding of what is being learned (Small 2014, loc 95).

Her avowed aim notwithstanding, the book is a confusing muddle. In the first instance, one would expect a competent teacher's guide to succinctly describe the structure of the subject I outline above, and to follow this by suggesting grade-appropriate examples of ways to develop children's understanding and fluency. But this is not the case. References to the fundamental concepts are scattered throughout the text, are mischaracterized, and may be incomplete. For example, definition (i) of the four for manipulating rational numbers is sloppily introduced as “the standard algorithm” in the chapter for Grade 5 students and definition (ii) is introduced in the Grade 6 material parenthetically—effectively as an aside—by stating “In general....” (Small, 2014, locs 1265, 1752).

In a self-congratulatory style typical of educators' literature her “Good Questions to Ask” at the end of each chapter are of the type that concerned Zwaagstra (2012). I cite six examples, three for Grade 5 and three for Grade 7.

Grade 5:

- *Tell students that you divided a whole number by $1/5$. Ask them to name three possible answers and three impossible answers and explain each.* [Hopefully students observe that the answer is always a multiple of 5]
- *Ask students to create a problem that is solved by dividing 4 by $1/4$.*
- *Ask students for a number just a little bit less than $-3/4$.*

⁷The University of Manitoba's Dr. Robert Craigen confirmed the accuracy of these exchanges.

Grade 7:

- Ask students to create a visual explanation why each is true;
 - $-4/3 \times (3/8) = -1/2$ (plus three others of this type)
 - Ask students to describe a situation by that might lead someone to divide -3.2 by 4
 - Ask students to create an argument for why $-2/3 \times (-3) = 2$ without just using a rule.
- (Small, 2014, locs 1662, 2196, italics in original)

A few questions such as the first two may be useful in developing students' confidence and understanding, but the book is almost completely devoid of the practice questions needed to develop fluency. The third is badly worded: does she mean magnitude, or is it displacement on the number line? The fourth could be a serious time waster. Finally, as the history of the rule of signs indicate, the last question suggests ignorance of the structure of arithmetic.

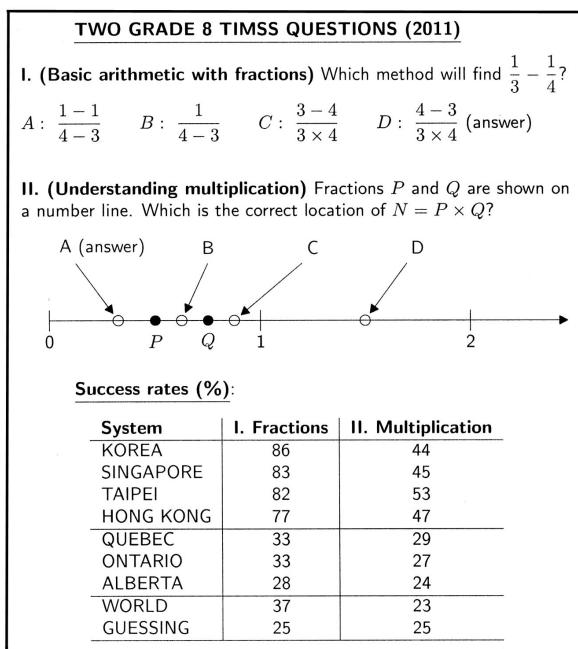
I conclude this commentary by noting the grave concerns about the quality of textbooks recently expressed by Annie Keeghan, who has “worked for over 20 years in educational publishing as a product developer, writer, and editor of curriculum materials for Grades K-8.”

There are only a small number of educational publishers left ... after buyouts and mergers of the 90's, publishers that all vie for a piece of a four billion dollar pie. ...The balance between the budgets for marketing and product development is growing further and further apart... Royalties are a thing of the past for most writers, and work-for-hire is the norm... The number of qualified writers is diminishing, and the those being contracted by developers and publishers often don't have the necessary skills or experience to produce a text worthy of the publisher's marketing claims...Sometimes the content in the student materials is so poor—steps omitted, unclear directions, concepts introduced when they're not developed until later in the text, distorted interpretations of math terms and applications — that it boggles the mind it got past a content editor. (Keeghan, 2012)

The situation in Ontario

The poor state of Ontario students' mathematical abilities is highlighted in a recent Trends in International Mathematics and Science Study (TIMSS) conducted by the U.S. National Center for Education Statistics. The adjacent diagram, compiled by Craigen (2013) from the 2011 study, compares scores obtained by Grade 8 students in four Asian countries with those from three Canadian provinces. The first of two four-choice questions tests a computational skill, and the second tests understanding. Note that, for these questions, 25% corresponds to random guessing. For the first, only 33% of Ontario's students obtained the correct answer; this contrasts with the four Asian countries' 77 to 86%. For the second question, at 27% the success rate is no better than random guessing. This in turn suggests that the

claims of Dr. Small and others that they are focussing on inculcating a “deep and rich understanding” are not substantiated. I infer that Ontario's mathematics teaching has deficiencies on all three



counts: facts, procedure and concepts.

I have not undertaken a detailed evaluation of Ontario's Ministry of Education mathematics curriculum or advisory literature, or of any of the approved text books, but such material as I have encountered leaves me deeply concerned. There is evidence of Wu's "fuzzy mathematics," advocacy of constructivist methods, as well as of questionable competence. Also I briefly examined my grandson's Grade 6 text: it is a mess similar to those described by Keeghan (2012).

I cite three examples from the Ministry's advisory literature. The first is a 2004 report on Ontario maths education in Grades 4 to 6 (MOE 2004). Although the title includes the words *Expert Panel*, all the experts are professors of education. Also, although it contains six pages of references, believing that instruction should be "based on problem solving and inquiry" it nevertheless echoes U.S. mathematics educators' fixation on constructivism. (MOE 2004, p. 59)

A major theme of the report is a script for a hypothetical problem-based Grade 5 lesson on division, accompanied by an extended commentary promoting the virtues of such an approach. The script states that the students have "a range of understandings and prior knowledge." The class is asked to ascertain the number of bags of marbles holding 23 each which must be purchased to fill a jar having space for 317. The teacher has not yet taught "a standard algorithm for division." She breaks up the class into small groups, giving each a jar and a bag of the marbles. The groups then work on the problem: "the class is noisy and there is a great deal of gesturing towards marbles and jars on the table, pouring of marbles into the jars, and writing of calculations and diagrams in mathematical books." The teacher circulates, asking the students for explanations and posing "probing" questions. However most of "the mathematics communication" is within and between the groups. After an extended period these groups are asked to write up their solutions, following which the teacher selects four to discuss, including that of one group which measured the heights of one bag and the jar. The teacher explains why the measurement approach does not work, and then explores the other three student solutions, one involving doubling up from 23, one involving "multiplying up," with the third being a form of long division. (MOE 2004, pp. 10-21).

The commentary describes "traditional" (direct?) instruction, citing education literature claiming that it is insufficient for generating a "deep understanding of mathematics for all," and that "students must construct or 'reinvent' mathematical ideas in order to understand them." It nevertheless acknowledges that "discovery instruction has ...proved generally insufficient for many children," and that what it calls "procedures based" teaching has a role. It also asserts that open-ended questions offer "the best framework for accessible problems for all students" (pp. 8-9, 11).

This exercise raises questions begging for answers. For example, why is it deemed desirable to provide marbles to students who have reached Grade 5? Is this a manifestation of the aversion to decontextualization identified by Anderson et al (2000)? Also—as seems to be implied by the script itself—having the students manipulate the marbles and jars may be little better than a distraction. Furthermore, the envisaged student collaboration may be fanciful. University of Toronto psychologist Ruth Pike has observed students attempting to solve problems in groups; she concluded that it is usually the brightest student who finds the solution on her own, with the other students passively copying that answer (Pike, 2014).

The report also makes contentious assertions, three being:

- (i) Historically [the standard algorithms] were created to be used for efficiency by a small group of "human calculators" when calculators were not yet invented... They were not designed to support the sense making that is now expected from students (underlined here for emphasis).

- (ii) Most adults learned rules and procedures by memorization first and then perhaps understanding later.
- (iii) Research findings to date indicate that unrestricted access to calculators does not adversely affect student performance in mathematics (MOE 2004, pp 12, 22, 29).

I suggest that expert teachers using “traditional methods” would take strong exception to the first two, and especially to the self-serving insinuation that “sense making” was not previously expected.

On the role of calculators, the report’s citation of literature notwithstanding, if their use impedes the development of automaticity and facility in basic mental arithmetic, caution should be exercised. In this respect a Singapore mathematics educator Koay Phong Lee (2006) provides a useful perspective. Note that Singapore’s students performed at the top of the TIMMS assessment cited above, and that the country’s Ministry of Education has banned the use of calculators in elementary school classrooms. Lee suggests ways in which they might be used to enhance understanding in these grades; but he does not recommend use of calculators when practising computational skills, a policy that is consistent with Price et al’s (2013) findings from fMRI experiments.

Finally I note that this report exhibits an example of mathematics educators’ penchant for muddying the mathematical waters: citing a recommendation in the educational literature, it lumps definitions and rules together, calling them “big ideas.” (MOE 2004, pp. 16, 19, 22)

My second example of the Ministry’s literature is “Problem-Based Learning in Mathematics” (Macnath et al, 2009); the document characterizes this method as “grounded in the ideals of constructivism and student-centred learning.” Its focus is an Ontario case history in which a Grade 6 teacher asked the class to work a “real-life” example: to develop a hockey division’s 80-game schedule in which 30% of the games have to be assigned to one group of players, 15% each to two other groups, and 40% to the fourth group. It notes that

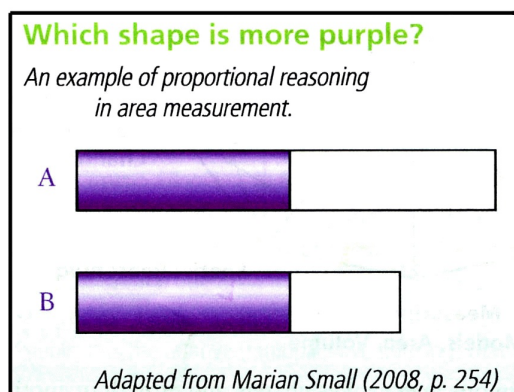
as multiplication and percentages had been covered in a full unit just two weeks prior, the teacher expected the students to [calculate the allocations] and move on to looking at travel distances. Instead all groups [of students] were stumped [sic!].”

After describing students’ fumbling attempts to solve the problem, the document observes that

the teacher was shocked at [the students’] inability to solve the problem. She repeatedly commented that the students had already been tested and received a C+ or higher. Interviews with the students revealed the root of the problem: Context matters. (Underlined here for emphasis)

Given the results of the abovementioned TIMMS test, and given the negative assessments of PBL such as that by Colliver (2000) I suggest that, contrary to the Macnath et al’s claim, the main source of failure is more likely to have been lack of adequate practice in the fundamentals, together the general tendency of problem-based and similar constructivist methods to fail. As Kirschner et al (2006) put it, “because students learn so little from a constructivist approach, most teachers who attempt to implement class-room based constructivist instruction end up providing students with considerable guidance.” Also, is MacNath et al’s bald assertion that “Context matters” yet another example of educators’ aversion to decomposition and decontextualization?

The third Ministry document I examined discusses what it terms “proportional reasoning” for grades K-12. (MOE 2012) The introduction includes the adjacent colour diagram adapted from one of Dr. Small’s textbooks. The accompanying text suggests that “Giving students non-numerical representations which require qualitative reasoning can evoke rich discussions about proportionality.” She has asserted that introduction of ambiguity “is a great way to teach math.” (Craig, 2016). With precision being a hallmark of mathematics, in the context of elementary student instruction, is not this fuzzification a prescription for confusion?



On pages 8-9 the document gives a list of suggested “mathematical tasks.” Among the questions suggested for “*Primary/Junior/Intermediate*” students are:

- How many different ways can you cut a sandwich in half?
- Eric says that $\frac{8}{8}$ is greater than $\frac{4}{4}$ because there are more pieces. Sylvia says $\frac{4}{4}$ because the pieces are bigger. What do you think?
- Create a growing or shrinking pattern using pennies or nickels starting with 20 cents.
- Research and report on probability situations expressed in fraction, decimal, and percent form.

These illustrate educators’ penchant for open-ended and vaguely worded questions, which could be both puzzling and time wasters. Consider the first: if the sandwich is made with rectangular bread slices, an elementary geometric construction allows one to cut it into equal halves having an infinite number of shapes. The second is yet another example of deliberate “fuzzy” ambiguity.

Among those suggested for “*Intermediate/Senior*” students are:

- Give an example of a linear relationship that is not proportional.
- Why might it be useful to report fuel efficiency as L/100 km? Would it be just as useful to report it as km/L?
- One cylinder has double the volume of another. How could the radii and heights be related?
- The method of “cross multiplication” is often used to solve a proportion problem ... Provide a good argument to show that cross multiplication is a valid method for solving a proportion problem.

The second question seems more appropriate to a psychology class than to mathematics. In the third, omission of the word “circular” needed to define the cylinder geometry is an example of an all-too-common tendency to be imprecise. As to the fourth, the phrasing suggests the writers’ confusion about the basic properties and definitions of fractions.

Also the use of the word “good” in the question’s formulation is yet another example of the fuzzification deplored by Wu (1996). Another fuzzy term frequently used in educators’ documents is the word *reasonable* to describe student responses; apparently the terms *right* and *wrong* are anathema.

This document also includes three samples of “annotated student solutions” to tests administered by the Ontario Government’s Education Quality and Accountability Office (EQAO), one each for Grades 3, 6 and 9. In each sample the annotations are detailed commentaries under four headings: *Assets*; *Wondering*; *Observation*; *Challenges*. The Grade 9 question and a student’s answer is given

in the adjacent box. The marker's evaluations make for interesting reading, I give an abbreviated version:

Assets: The student successfully sets up a table to compare dog weight and correctly lists three equivalent ratios... indicating proportional reasoning (multiplicative thinking) She/he adequately explains the halving strategy but does not clearly explain that she/he is finding the dosage for a 25 kg dog.

Wondering: When filling out the table, was the student using additive thinking, filling it in vertically by following two separate patterns? ... or seeing the relationship between horizontal number pairs? How did the student calculate 22? What would the student have done if she/he used the correct difference of 25...since it is odd?

Observation: While the student incorrectly calculates distance (sic!) between 50 and 75, using 22 rather than 25, she/he successfully applies the strategy of using his/her halfway point of 11... to calculate the dosage for a 25 kg dog. The strategy reveals an understanding of proportional reasoning and multiplicative thinking involving halves.

Challenges: The student uses additive reasoning to calculate the weight for a 24 kg dog, reasoning that a dog that is 1kg less than 25 kg would need 1ml less of medication.. This could indicate that proportional reasoning is still fragile when numbers are less friendly (sic!) and do not conform to halving, doubling or tripling.

Clarence's Quandary

Clarence works at a veterinarian's office. He needs to give a dose of medicine to a 24 kg dog. The recommended dosage for a dog that weighs 10 kg is 25 mL. Determine the dose Clarence should give to the 24 kg dog if the rate remains the same. Show your work.

Dog	med
10k	25ml
20k	50ml
30	75ml

50ml - 75ml
The amount of number between
that would be 22
half of 22 is 11 between
50 and 75 is 61 but would use sixty because the dog is
24kg.
∴ the dose of medicine
to a 24kg dog is
60ml

I make two inferences. First, the outlined marking scheme suggests that Ministry officials adhere to the constructivists' take on evaluation criticised by Anderson et al (2000), in which the emphasis is on process rather than on product. More generally, this assessment—riddled as it is with educators' jargon—raises questions about both the appropriateness and the competence of the EQAO marking procedures. For example, why is there no reference to curriculum expectations? What are the qualifications of the markers? Have they any mathematical training? How much money is squandered on this type of exercise?

The second inference is more serious. If the script in the hypothetical marble-counting class described in MOE(2004) is representative of Ontario's curriculum expectations, this Grade 9 student uses a method that is appropriate to Grade 5, using faulty reasoning to boot. If this level of competence is deemed to be sufficiently representative to include in a Ministry publication, is not this tantamount to admission of failure of the Ministry's policies?

Finally I note that this document commits the elementary blunder of defining rational numbers as "numbers that can be expressed as fractions."(MOE 2012, p.7) It is not simply an editing error,

it is the definition given in the official Grade K-8 curriculum. (MOE 2005, p.131) . The MOE (2005, p.127) definition of *irrational number* is also wrong; it is described as “A number that cannot be represented as a terminating or repeating decimal.”(ibid, p.127) This is a property of the numeral system and not a definition; the correct definition is one that is *not* rational. Competent dictionaries give both definitions correctly. This and other items I cite lead me to question the competence of the individuals who developed Ontario’s most recent mathematics curriculum (MOE, 2005).

Conclusion: thoughts about reform

I believe that reform of Ontario’s public school mathematics programme is required: issues in the curriculum, in textbook design, in teaching practices and in teacher training all need to be addressed. Since it is widely acknowledged that mathematics educators’ approach to mathematics can differ considerably from that of mathematicians, the latter must be involved. (Sultan and Artzt, 2005) Furthermore it may prove helpful to consult cognitive psychologists having insights into the way children develop numeracy.

TOPIC	GRADE:	1	2	3	4	5	6	7	8
Whole Number Meaning		■	■	■	■	■			
Whole Number Operations		■	■	■	■	■			
Measurement Units		□	■	■	■	■	■	■	
Common Fractions				□	■	■	■		
Equations & Formulas				□	■	■	■	■	■
Data Representation & Analysis				□	□	■	■	■	□
2-D Geometry: Basics				□	■	■	■	■	■
Polygons & Circles					■	■	■	■	■
Perimeter, Area & Volume					■	■	■	■	□
Rounding & Significant Figures					■	■			
Estimating Computations					■	■	■		
Properties of Whole Number Operations					□	■			
Estimating Quantity & Size					□	□			
Decimal Fractions					■	■	■		
Relationship of Common & Decimal Fractions					■	■	■		
Properties of Common & Decimal Fractions						■	■		
Percentages						■	■		
Proportionality Concepts						■	■	■	□
Proportionality Problems						■	■	■	■
2-D Coordinate Geometry						□	□	■	■
Geometry: Transformations							■	■	■
Negative Numbers, Integers & Their Properties							□	■	
Number Theory								■	□
Exponents, Roots & Radicals								■	■
Exponents & Orders of Magnitude								□	□
Measurement Estimation & Errors								□	
Constructions w/ Straightedge & Compass								■	□
3-D Geometry								■	■
Congruence & Similarity									■
Rational Numbers & Their Properties									□
Patterns, Relations & Functions									□
Slope & Trigonometry									□
Number of topics covered by at least 67% of the A+ countries		3	3	7	15	20	17	16	18
Number of additional topics intended by A+ countries to complete a typical curriculum at each grade level		2	6	5	1	1	3	6	3

□ – intended by 67% of the A+ countries ■ – intended by 83% of the A+ countries ■ – intended by 100% of the A+ countries

Figure 1 “Mathematics topics intended at each grade by at least two-thirds of A+ countries. Note that topics are introduced and sustained in a coherent fashion, producing a clear upper-triangular structure.” (Schmidt et al, 2002) The A+ countries are: The Czech Republic, Flemish Belgium, Japan, Singapore and South Korea.

Since the Ontario's policies, practices, and levels of student achievement are apparently similar to those in the USA, two recent reviews of U.S. deficiencies should give useful guides for reform. The first, by Schmidt et al (2002), compares the grades K-8 curricula of 21 U.S. states with those for a selection of high performing countries. The lead author, William Schmidt, of the University of Minnesota's College of Education was, at the time of writing, Director of the U.S. National Research Center producing TIMMS. The second source is the U.S. Department of Education's *Final Report of the National Mathematics Advisory Panel* (NMAP, 2008); this 117 page report asserts that "the delivery system in mathematics education ... is broken."(p.xiii) I quote a few comments and recommendations proposed by these reviews.

TOPIC	GRADE:	1	2	3	4	5	6	7	8
Whole Number Meaning		■	■	■	■	■	□		
Whole Number Operations		■	■	■	■	■	□		
Measurement Units		■	■	■	■	■	■	■	■
Common Fractions		□	■	■	■	■	■	□	□
Equations & Formulas		□	□	■	■	■	■	■	■
Data Representation & Analysis		■	■	■	■	■	■	■	■
2-D Geometry: Basics		■	■	■	■	■	■	■	■
Polygons & Circles		■	■	■	■	■	■	■	■
Perimeter, Area & Volume			□	□	□	■	■	■	■
Rounding & Significant Figures									
Estimating Computations		□	□	■	■	■	■	■	■
Properties of Whole Number Operations		□	□	□	□				
Estimating Quantity & Size				□					
Decimal Fractions				□	■	■	■	□	□
Relationship of Common & Decimal Fractions					□	□	□		
Properties of Common & Decimal Fractions									
Percentages						□	■	■	□
Proportionality Concepts							■	□	
Proportionality Problems							■	■	■
2-D Coordinate Geometry				□	■	□	□	□	■
Geometry: Transformations		■	■	■	■	■	■	■	■
Negative Numbers, Integers & Their Properties							□	■	□
Number Theory						■	□	□	□
Exponents, Roots & Radicals							□	□	■
Exponents & Orders of Magnitude								□	□
Measurement Estimation & Errors		□	□	■	□	■	■	■	□
Constructions w/ Straightedge & Compass									
3-D Geometry		■	■	■	■	■	■	■	■
Congruence & Similarity						□	■	■	□
Rational Numbers & Their Properties							■	■	□
Patterns, Relations & Functions		■	■	■	■	■	■	■	■
Slope & Trigonometry									
Number of topics covered by at least 67% of the states		14	15	18	18	20	25	23	22
Number of additional topics intended by states to complete a typical curriculum at each grade level		8	8	7	8	8	5	6	6
□ – intended by 67% of the states ■ – intended by 83% of the states ■ – intended by 100% of the states									

Figure 2 “Mathematics topics intended at each grade by at least two thirds of 21 U.S. states. Note that topics are introduced and sustained in a way that produces no visible structure.” (Schmidt et al, 2002)

Figure 1 gives Schmidt and his colleagues' analysis of the curricula for their high-performing (A+) countries. They observe that

These data indicate that across the A+ countries there is a generally agreed-upon set of mathematical topics—those related to whole numbers and measurement—that serve as the foundations of mathematics understanding. They constitute the fundamental mathematics knowledge that students

are meant to master during grades one to five. Future mathematics learning builds on this foundation. At the middle and upper grades, new and more sophisticated topics are added—and, significantly, *the foundational topics disappear from the curriculum.* (italics in original, Schmidt et al., 2002)

In marked contrast, their analysis for 21 U.S. states, given in Figure 2, shows “repetition and incoherence.” Some of the differences are striking; for example, all states introduce in Grade 1 the three topics *Data Representation & Analysis, Polygons & Circles* and *3-D Geometry*.

They also infer that, while—owing to cultural differences—one cannot expect to import wholesale the practices of other countries, nevertheless

Differences in achievement from country to country are related to what is taught in different countries. In other words, this is not primarily a matter of demographic variables... *the curriculum itself—what is taught—makes a huge difference.* (Italics in original, underlined for emphasis.)

In a comment on professional development attached to Schmidt et al’s (2002) analysis, Michigan State University education professor Mary Kennedy observes that

The more successful [professional development] programs provided knowledge that tended *not* to be purely about subject matter—that is, they were not courses in mathematics—but instead were about *how students learn* that subject-matter (italics in original).

Yet such development can also be ineffective: in this respect Schmidt et al (2002) observe:

In the U.S., the correlation between textbook coverage and what teachers teach is 0.95 (which is comparable to other countries). If we pretend the textbook doesn’t exist—and conduct professional development in ways that assume teachers will implement an entirely different approach to the content than the texts take—believe me, the text books will win. (Schmidt et al, 2002, underlined here for emphasis)

The U.S. Department of Education’s report (NMAP, 2008) raises the concerns expressed by Schmidt et al (2002), as well as many of those I express in this essay. I give a selection of the panel’s findings, recommendations and comments.

By the term *proficiency*, the Panel means that students should understand key concepts, achieve automaticity as appropriate, ... develop flexible, accurate and automatic execution of the standard algorithms, and use these competencies to solve problems. (p. xvii)

Calculators should not be used on test items designed to assess computational facility. (p.xxv)

There seem to be two major differences between the curricula in top-performing countries and those in the U.S.—in the number of mathematical concepts or topics presented at each grade level and in the expectations for learning. U.S. curricula typically include many topics at each grade level, with each receiving relatively limited development, while top-performing countries present fewer topics at each grade level but in greater depth. In addition, U.S. curricula generally review and extend at successive grade levels many (if not most) topics already presented at earlier grade levels, while the top-performing countries are more likely to expect closure after exposure, development and refinement of a particular topic. These critical differences distinguish a spiral curriculum ... from one built on developing proficiency—a curriculum that expects proficiency in the topics that are presented before

more complex or difficult topics are introduced (pp. 20-21)

Debates regarding the relative importance of conceptual skills, procedural skills and the commitment of addition, subtraction, multiplication and division facts to long-term memory are misguided. These capabilities are mutually supportive...(p. 26)

Few curricula in the United States provide sufficient practice to ensure fast and efficient solving of basic fact combinations and execution of the standard algorithms[furthermore] many U.S. middle school students do not understand the concept of mathematical equality. (p.26)

When mathematicians reviewed recently published middle and high school text books, they identified many errors and a large number of ambiguous and confusing statements and problems. (p.55)

U.S. mathematics textbooks are extremely long... elementary school textbooks sometimes exceed 700 pages.... [textbooks are] much shorter in many nations with higher mathematics achievement than in the U.S. states.... All parties involved in the publication adoption of textbooks should strive for more compact and more coherent mathematics texts for use by students in Grades K-8 and beyond. (p.55)

Fluency With Whole Numbers

- 1) By the end of Grade 3, students should be proficient with the addition and subtraction of whole numbers.
- 2) By the end of Grade 5, students should be proficient with multiplication and division of whole numbers.

Fluency With Fractions

- 1) By the end of Grade 4, students should be able to identify and represent fractions and decimals, and compare them on a number line or with other common representations of fractions and decimals.
- 2) By the end of Grade 5, students should be proficient with comparing fractions and decimals and common percent, and with the addition and subtraction of fractions and decimals.
- 3) By the end of Grade 6, students should be proficient with multiplication and division of fractions and decimals.
- 4) By the end of Grade 6, students should be proficient with all operations involving positive and negative integers.
- 5) By the end of Grade 7, students should be proficient with all operations involving positive and negative fractions.
- 6) By the end of Grade 7, students should be able to solve problems involving percent, ratio, and rate and extend this work to proportionality.

Geometry and Measurement

- 1) By the end of Grade 5, students should be able to solve problems involving perimeter and area of triangles and all quadrilaterals having at least one pair of parallel sides (i.e., trapezoids).
- 2) By the end of Grade 6, students should be able to analyze the properties of two-dimensional shapes and solve problems involving perimeter and area, and analyze the properties of three-dimensional shapes and solve problems involving surface area and volume.
- 3) By the end of Grade 7, students should be familiar with the relationship between similar triangles and the concept of the slope of a line.

Figure 3. The U.S. National Mathematics Advisory Panel's "Benchmarks for the Critical Foundations." (NMAP 2008, p. 20)

I suggest that all of these issues apply to Ontario. In this respect Figure 3 gives the NMAP report's suggested curriculum goals for the grades K-7. Stokke's (2015) commentary recommends this as a guide for improvement of Canadian curricula.

To conclude, I recommend that the Ontario Government convene a review panel equivalent to the U.S. National Mathematics Advisory Panel, and having membership from universities, colleges, schools, and from concerned professions. This panel should examine all aspects of the problem, including curriculum, textbooks, and teacher training. Furthermore, the role and effectiveness of EQAO procedures needs scrutiny. It should also be useful to undertake an analysis of the most recent curriculum (MOE 2005) equivalent to that in Figure 1.

ACKNOWLEDGEMENTS.

I am grateful for the advice and encouragement that I have received from many individuals during preparation of this essay; this includes teachers, professional engineers, and academics. I particularly wish to thank mathematicians Ed Barbeau of the University of Toronto and Robert Craigen of the University of Manitoba, both of whom are especially cognizant of the issues.

REFERENCES

- Anderson, J. R., L.M. Ryder, and H.A. Simon. 2000. Applications and Misapplications of Cognitive Psychology to Mathematics Education. *Texas Educational Review* (Summer).
<http://www.act-r.psy.cmu.edu/papers/misapplied.html>
- Atherton, J. S. 2013. Learning and Teaching; What Works Best.
http://www.learningandteaching.info/teaching/what_works.htm [retrieved 5-01-16]
- Audi, R. (Editor) 1999. *The Cambridge Dictionary of Philosophy*. Cambridge: Cambridge University Press. See pp.78, 149.
- Barbash, S. 2012. Clear Teaching: With Direct Instruction, Siegfried Engelmann Discovered a Better Way of Teaching. Educational Consumers Foundation.
http://www.education-consumers.org/pdf/CT_111811.pdf
- Carnine, D. 2000. Why Education Experts Resist Effective Practices. Fordham Foundation Report.
<http://www.wrightslaw.com/info/teach.profession.carnine.pdf>
- Christodoulou, D. 2014. *Seven Myths About Education*. London: Routledge (E-book).
- Colliver, J. A. 2000. Effectiveness of Problem-based Learning Curricula: Research and Theory. *Academic Medicine*. **75** (3) March, 259-266.
- Courant, R., and H. Robbins. 1981. *What Is Mathematics?: an Elementary Approach to Ideas and Methods*. Oxford: Oxford University Press.
- Craigen, R. 2013. Why does 'fuzzy math' persist? It's a bonanza for consultants, publishers. Globe and Mail, Dec.4.
<http://www.theglobeandmail.com/news/national/education/why-does-fuzzy-math-persist-its-a-bonanza-for-consultants-publishers/article15763158/>
- . 2016. Private communication.
- Dziwak, K. 2014. Tackling Student Literacy and Numeracy Underpreparedness in Ontario Colleges: Access, quality and funding. *College Quarterly*. **17**(2).
www.collegequarterly.ca/2014-vol17-num02-spring/dziwak.html
- Hirsch, E. D. 1999. *The Schools We Need And Why We Don't Have Them*. New York: Anchor Books. (E-book)
- Keeghan, A. 2012. Afraid of Your Child's math Textbook? You Should Be.
<http://www.bradthiessen.com/html5/m340/11b%20Afraid%20of%20Your%20Child's%20Math%20Textbook.pdf>

- Kershaw, A. 2010. Big drop in math skills of entering students. *University Affairs*. September 13.
<http://www.universityaffairs.ca/news/news-article/big-drop-in-math-skills-of-entering-students.aspx>.
 Accessed February 16, 2016
- Kirschner, P. A., J. Sweller and R. E. Clark. 2006. Why Minimal Guidance Instruction Does Not Work: An Analysis of the Failure of Constructivist, Discovery, Problem-based, Experiential, and Inquiry-Based Teaching. *Educational Psychologist*. **41**(2), pp. 75-86.
- Klein, D. 2003. A Brief History of American K-12 Mathematics Education in the 20th Century.
<http://www.csun.edu/~vcnth00m/AHistory.html>. Accessed 6 February 2016
- Klein, D. and R. J. Milgram. Undated. The Role of Long Division in the K-12 Curriculum.
<http://www.csun.edu/~vcnth00m/longdivision.pdf> Accessed February 16, 2016
- Lee, K. P. 2006. Calculator Use in Primary School Mathematics: A Singapore Perspective. *The Mathematics Educator*. **9** (2), 97-111.
- Leinwand, S. 1994. It's Time to Abandon Computational Algorithms. *Educational Week*, February 9.
<http://www.edweek.org/ew/articles/1994/02/09/20lein.h13.html>. Accessed February 16, 2016
- Macnath, S., J. Wallace, and X. Chi. 2009. Problem-Based Learning in Mathematics. A Tool for Developing Students' Conceptual Knowledge. Ontario Ministry of Education, Literacy and Numeracy Secretariat Research Monograph #22, November.
- MOE (Ontario Ministry of Education) 2004. Teaching and Learning in Mathematics. The Report of the Expert Panel on Mathematics in Grades 4 to 6 in Ontario.
<https://www.edu.gov.on.ca/eng/document/reports/numeracy/panel/numeracy.pdf>
- MOE 2005. The Ontario Curriculum Grades 1-8 Mathematics.
<https://www.edu.gov.on.ca/eng/curriculum/elementary/math18curr.pdf>
- MOE. 2012. Paying Attention to Proportional Reasoning, K-12.
<https://www.edu.gov.on.ca/eng/teachers/studentsuccess/ProportionReason.pdf>
- NMAP. 2008. The Final Report of the National Mathematics Advisory Panel. U.S. Department of Education.
<http://www2.ed.gov/about/bdscomm/list/mathpanel/report/final-report.pdf>
- Nrich Maths. 2016. The History of Negative Numbers. Accessed January 28, 2016.
<https://nrich.maths.org/5961>.
- Pike, R. 2014. Private communication.
- Price, G.R, M.M. Mazzocco and D. Ansari. 2013. Why Mental Arithmetic Counts: Brain Activation during Single Digit Arithmetic Predicts High School Math Scores. *The Journal of Neuroscience*. **33**(1): 156-163.
<http://www.jneurosci.org/content/33/1/156.full>
- Schmidt, W., et al. 2002. A Coherent Curriculum: The Case of Mathematics. *American Educator*. Summer, 1-17.
<https://www.aft.org/sites/default/files/periodicals/curriculum.pdf>
- . 2008. What's Missing from Math Standards? *American Educator*. Spring, 22-24
<http://www.aft.org/sites/default/files/periodicals/schmidt.pdf>
- Siegler, R. et al. 2012. Early Predictors of High School Mathematics Achievement. *Psychological Science*. **23** (7), 691-697.
- Small, M. 2014. *Uncomplicating FRACTIONS to Meet Common Core Standards in Math, K-7*. New York: Teachers College Press (Nelson Education: E-book)
- Stokke, A. 2015. What to Do about Canada's Declining Math Scores. C.D. Howe Institute. Commentary No. 427. May.
- Sultan, A., and A. F. Artzt. 2005. Mathematicians Are from Mars, Math Educators Are from Venus: The Story of a Successful Collaboration. *Notices of the AMS*. **52** (1), 48-53.
- Wikipedia 1. 2016. Trends in International Mathematics and Science study. Accessed January 27, 2016
https://en.wikipedia.org/wiki/Trends_in_International_Mathematics_and_Science_Study

- Willingham, D. 2009. Is It True That Some People Just Can't Do Math? *American Educator*. Winter 2009-10, 14-19,39.
<https://www.aft.org/sites/default/files/periodicals/willingham.pdf>
- Wilson, W. S. 2011. In Defense of Mathematical Foundations. *Educational Leadership*. March: 70-73.
<http://www.math.jhu.edu/~wsw/ED/ramblepublished.pdf>.
- Wu, H. 1996. The Mathematician and Mathematics Education Reform. *Notices of the American Mathematical Society*. **43**(12), 1531-37. Available on internet in updated form as "The mathematics education reform: What is it and why should you care?,"
<https://math.berkeley.edu/~wu/reform3.pdf>, and <https://math.berkeley.edu/~wu/wu1999.pdf>
- . 1999. Basic Skills Versus Conceptual Understanding: A Bogus Dichotomy in Mathematics Education. *American Educator/American Federation of Teachers*. Fall. 2-7.
- . 2011a. The Mis-Education of Mathematics Teachers. *Notices of the AMS*. **58** (3), 372-384. (March)
- . 2011b. The Mathematics K-12 Teachers Need to Know. (August)
<https://math.berkeley.edu/~wu/Schollmathematics1.pdf>
- Zwaagstra, M. 2012. How to make math education worse in N. S. *The Chronicle Herald*, Oct. 3
<http://thechronicleherald.ca/opinion/142943-how-to-make-math-education-worse-in-n.s>