# Opinion Piece: Mathematics as a science of the real world: Aristotelian realist philosophy of mathematics

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## Introduction

Mathematics can seem to scientists and science teachers a confusing pile of formulas, methods and calculations, not a genuine science with a subject matter of its own. Aristotelian realist philosophy of mathematics holds on the contrary that mathematics is as much a science of an aspect of reality – physical reality and any other reality there might be – as physics, biology and sociology. Mathematics, it says, is about the quantitative aspects of the world (such as ratios) and the structural ones (such as symmetry). Those properties are as real as mass or biodiversity.

That view of mathematics has important implications for both mathematics teaching and science teaching. For mathematics teaching, it suggests the need for more attention to mathematical modelling and less to the more internal techniques of calculation and manipulation of formulas. For science teaching, it suggests an approach that highlights an awareness of mathematical properties in science, rather than sweeping mathematics under the carpet as maths teachers' business.

## Aristotelian realism versus Platonism and nominalism

What is mathematics about? We know what biology is about; it's about living things. Or more exactly, the living aspects of living things – the motion of a cat thrown out of a window is a matter for physics, but its physiology is a question of biology. Oceanography is about oceans; sociology is about human behaviour in the mass long-term; and so on. When all the sciences and their subject matters are laid out, is there any aspect of reality left over for mathematics to be about? That is the basic question in the philosophy of mathematics.

The field of philosophy of mathematics is mainly occupied by two longstanding traditions, Platonism and nominalism, which have opposite answers to this question. Both answers are problematic and Aristotelian realism offers a "third way" that is more closely aligned to how mathematics works in science (Franklin, 2014a, Jacquette, 2014, Gillies, 2015; quick introduction in Franklin, 2014b, survey in Franklin, 2021).

According to Platonism – the name deriving from Plato's view that eternal realities exist in a non-physical realm – the "objects" named in mathematical discourse, such as numbers, sets and functions, are fully real but not part of our (physical) world. They exist in a Platonic heaven of "abstract objects", eternal, non-physical and non-causal.

What makes Platonism attractive is our solid feeling that mathematics discovers truths about a pre-existing terrain. Take the subtleties of the distribution of primes. Some numbers are prime, some not. A dozen eggs can be arranged in cartons of $6\times2$ or $3\times4$, but eggs are not sold in lots of 11 or 13 because there is no neat way of organising 11 or 13 of them into an egg carton: 11 and 13, unlike 12, are prime, and primes cannot be formed by multiplying two smaller numbers. The idea is very easy to grasp. But there is a lot to discover about it.

It is found that the way in which the primes are distributed among numbers involves a complex interplay of pattern and irregularity. On the small scale, the latter is most evident: there are long stretches without any primes at all – indefinitely long stretches, in fact. (For example, there are none between 113 and 127.) At the same time, it is widely believed that there are infinitely many "prime pairs"; that is, pairs of numbers only two apart that are both prime, such as 41 and 43.

When we turn to the large scale, the impression of disorder fades and a pattern starts to emerge after all. Primes become gradually less dense as one counts up: the density of primes around a large number is inversely proportional to its order of magnitude. The density of primes around a trillion ($10^{12}$), for example, is about half what it is around a million ($10^{6}$). It's all "out there", and it appears to be about a realm of abstract numbers. It is not about anything physical, but neither did we make it up. We have a sense of a non-physical reality, in some sense not our choice and waiting for us to discover it.

Platonism is also suggested by role of idealisation in mathematical applications in science. Aristotle himself describes Protagoras "refuting" the geometers by pointing out that a hoop touches a straight line at more than one point, unlike the perfect circles that geometers study. (Metaphysics 997b35-998a4). The perfect circles, it seems, must live in a realm other than the physical one we sense and measure.

Nominalism has been Platonism's main rival in the philosophy of mathematics. Taking its name from *nomina*, meaning names, it says that mathematical entities do not exist at all and the words apparently referring to them are "mere names". Only physical objects really exist, and the mathematical language that appears to refer to mathematical objects is just a language of science, or manipulation of formal symbols, or fictions, or methods of deriving one contentful truth from another (nominalism divides into several schools such as formalism, logicism and fictionalism, the differences among which are not important here).

An often implicit version of nominalism rife among scientists thinks of mathematics as a toolkit of methods, formulas, tables of Laplace transforms and the like, or as a ”theoretical juice extractor” for getting predictions from theories, but not itself actually about any aspect of reality. Sadly that view, understandable enough in physicists and engineers, is reinforced by the rule-based style of teaching mathematics that often fills the traditional school curriculum. (”Minus times minus equals plus/ The reason for this we need not discuss.”). That reinforces a view of mathematics as detached from real science. Einstein, though one of the most mathematical of physicists, was typical of them in claiming a divorce between mathematics and physical reality, saying “As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality” (Einstein, 1954, 233).

Nominalism has two problems. First, it fails to explain our felt sense of exploring a pre-existing territory, as in the distribution of primes. Secondly, it fails to answer the problem raised in the celebrated essay of 1960 by the physicist Eugene Wigner, of the “unreasonable effectiveness of mathematics in the natural sciences” (Wigner, 1960). How could a mere language deliver such extraordinarily effective and unexpected results in so many natural sciences?

Aristotelian realism offers to break the deadlock between Platonism and nominalism. It holds that mathematics *is* a contentful science of realities (as the Platonist says) but that those realities – at least many of them – are literally part of, or realised in, physical reality (Franklin, 2014a).

But Aristotelian realism is not just an answer to an existing philosophical debate. If we just look at how mathematics works (especially applied mathematics), free of philosophical preconceptions, it is possible to see what properties of reality are mathematical (rather than physical, biological and so on) and form the true subject-matter of mathematics. Imagine the Earth in the Jurassic Era, before there were humans to think mathematically and write formulas.

An artist’s impression can be seen here.

There were dinosaurs large and small, trees, volcanoes, flowing rivers and winds …Were there, in that world, any properties that we would recognize as being of a mathematical nature (to speak as non-committally as possible)? That is, were there, among the properties of the real things in that physical world, some that we would naturally recognise as mathematical (over and above physical, chemical and biological properties)?

There were many such properties. Symmetry, for one. Like most higher animals, the dinosaurs had approximate bilateral symmetry. The trees and volcanoes had an approximate circular symmetry with random elements—seen from above, they look much the same when rotated around their axis. But symmetry, whether exact or approximate, is a property that is not exactly physical. Non-physical things can have symmetry; arguments and palindromes, for example, have symmetry if the last half repeats the first half in the opposite order. Symmetry is an uncontroversially mathematical property, and a major branch of pure mathematics—group theory—is devoted to classifying its kinds. When symmetry is realized in physical things, it is often very obvious to perception—even animals as primitive as bees can perceive symmetry. Symmetry, like other mathematical properties, can have causal powers, unlike abstracta as conceived by Platonists.

Another mathematical property, which like symmetry is realizable in many sorts of physical things, is ratio. The height of a big dinosaur stands in a certain ratio to the height of a small dinosaur. The ratio of their volumes is different—in fact, the ratio of their volumes is much greater than the ratio of their heights, which is what makes big dinosaurs ungainly and small ones sprightly. A given ratio is something that can be the relation between two heights, or two volumes, or two time-intervals; a ratio is just what those relations between different kinds of physical entities share, and is thus a more mathematical property than the physical lengths, volumes, and times themselves. Ratio is what we measure when we determine how a length (or volume, or time, etc) relates to an arbitrarily chosen unit (Michell, 1994). It is one of the basic kinds of number.

Properties of reality like symmetry and ratio and others (such as flows, order relations, continuity and discreteness, alternation, linearity, feedback, network topology)—which are measurable, perceivable and causal, like other scientific properties—must be the subject of some science. That science is mathematics (or at least part of mathematics). Aristotelian realist philosophy of mathematics has consequences for both mathematics teaching and science teaching.

## Mathematics teaching: Mathematical modelling

Mathematics teachers cannot allow themselves be trapped into a servicing role, as if their task is to supply students with methods which will help when the student comes to subjects that really grapple with the world, like physics. (Applied) mathematics is about the real world too. School and college mathematics education needs to include a component of mathematical modelling, the process (as Aristotelian realists would put it) of finding the mathematical structure of the real world that is relevant to solving some problem.

A simple problem is: could a water shortage in a location like Los Angeles or Adelaide be alleviated by towing an iceberg from Antarctica? Anyone – or any class of students divided into groups – can usefully brainstorm ideas on what quantitative information is needed to address the problem. (How big are icebergs down there? How long would it take to tow them? How much would melt on the way? Would the amount left be a worthwhile proportion of the city's water consumption? What can be done with them when they arrive?) Given an hour, groups can report out to a class on a plan of attack. Given a week, they can do research and write a respectable feasibility study (Banks, 2013, ch. 6). A report lays out the mathematical structure of the case, with a view to making recommendations.

As the example shows, the teaching of mathematical modelling is very different in style as well as content from pure mathematics. Where pure mathematical skills are usually assessed by exams in which individuals solve short discrete puzzles which they get right or wrong, modelling is most naturally done collaboratively over a considerable time period, using outside research, and communicating via a written or oral joint report. That is similar to how mathematics is really done in industrial settings.

As in pure mathematics, mathematical modelling is not all bare hands from a standing start, as in the icebergs problem. Education needs to provide tools to think with. Population modelling has proved a good test bed for explaining basic mathematical models like exponential growth.

Example: Suppose a lake has some lily pads in it and suppose each pad replicates itself once a week. If it takes half a year for half the lake to become covered in lily pads, how long will it take for the entire lake to be covered? (Vandermeer and Goldberg, *Population Ecology*, 3)
Answer: 1 week.
Conclusion: Exponential growth can pick up speed.

Exponential growth has the typical "rising graph" shape, with typical formula $P = Aa^t$ (with a > 1) In the lily-pad example, $a = 2$ (where time $t$ is in weeks).

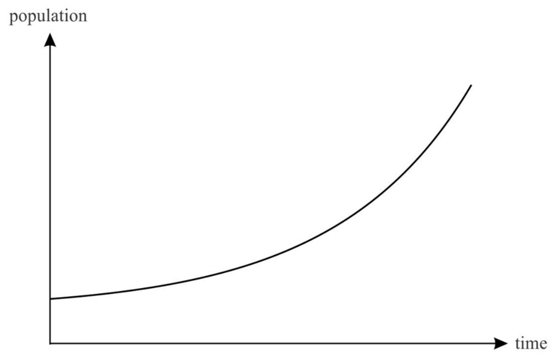


Fig 1: Graph of exponential growth of population against time

A model such as this formula means a mathematical description of the (possible) structure of a situation (such as how a population grows over time). To say that it is a possible structure, or a tool to think with, is not to say that actual populations must fit the model. Given observations of a real population, some inference is needed to see if its growth approximately fits an exponential shape.

An example is: Observations of the number of aphids on a typical corn plant in a field are given in the table below. Action must be taken when the number of aphids per plant reaches 40. At what date should that be predicted? (Vandermeer and Goldberg, 6-7)

Table 1 Number of Aphids Observed per Plant in a Milpa (Corn and Beans) in the Highlands of Guatemala (Morales, 1998)

If the growth is close to exponential, then the logarithms of the number of aphids (last column) should be approximately on a straight line. We can then plot the logarithms against time, check (by eye or by statistical software) if the five points are approximately on a straight line. In fact they are. We can then project forward to estimate the approximate date when 40 aphids per plant will be reached.

What about the world population of humans? In the 1960s there were many alarmist predictions that exponential growth of the world's population would lead to disaster in a few decades. But the world's birth rate has fallen dramatically since that time. The UN's predictions for world population out to 2100 can be seen here. It is a good lesson on the need to fit models to data with care.

Excellent educational resources for mathematical modelling are available from COMAP(The Consortium for Mathematics and Its Applications). Since 1985 they have run an undergraduate Mathematical Contest in Modeling and have recently begun one for high schools.

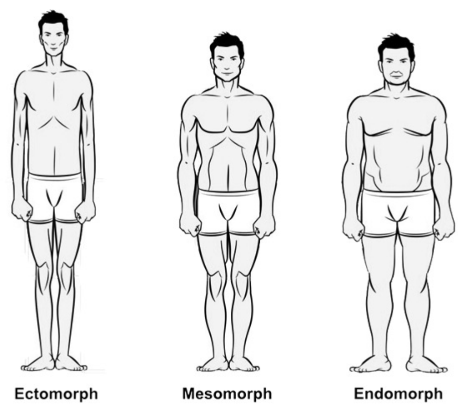


Fig 2. The classic body types: different height-

width proportions are easily perceived (Image credit: Granito Diaz, Wikimedia file Body-types.jpg)

## Science teaching: Awareness of mathematical properties

Philosophy of science and science teaching are incomplete without some sense of the difference between physical properties and mathematical properties. Physical properties, like mass, colour, being gold and being a possum, are observable realities which must be studied, at least in part, observationally. Mathematical properties, like ratio, symmetry, alternation and randomness, are also observational realities, but they are studied by the a priori methods of mathematics. Appreciating the mathematical aspects of physical reality is a matter of calling attention to mathematical commonalities between different physical situations.

Take as an example perhaps the most basic purely mathematical property, ratio or proportion, such as the proportion of length to height of a page or computer screen or image (the "aspect ratio"). It is an easily observable property of reality. Our visual systems are well set up to perceive immediately differences in ratio such as those in the classic body types (Fig 2)

But ratios are much more mathematical and abstract than lengths themselves. A given ratio can exist between two lengths, or two masses, or two time intervals. The truths about ratios are provable truths of mathematics, such as the very ancient Greek result of the incommensurability of the ratio of the diagonal and side of a square.

Despite their abstractness, ratios have practical scientific consequences, as we saw above in the case of the explanation of why small animals can scurry but large ones lumber. Another place where the scientific role of proportion is easily appreciated comes from the central role in scientific education played by the laws of proportion that formed a key part of the Scientific Revolution. In the high period of the Scientific Revolution, a number of laws were discovered which were mathematical in one sense, in that they ascribed to nature simple formulas – indeed, formulas in general of simple proportion. They were not purely mathematical, in the sense that (to the disappointment of some) they are not derivable solely from mathematical axioms. They needed some input – however small – of empirical and observationally-derived fact.

A list of the Scientific Revolution's laws of proportion, with approximate dates, includes:

- Kepler's Second Law: The area swept out by a radius from the sun to a planet is proportional to the time taken (1609).
- Snell's Law: When light is refracted at a surface, the sine of the angle of refraction is proportional to the sine of the angle of incidence (1602, 1621, 1637).
- Galileo's Law of Uniform Acceleration: The speed of a heavy body falling from rest is proportional to the time from dropping (1638).
- Pascal's Law: The pressure in an incompressible fluid is proportional to depth (1647).
- Hooke's Law: The extension of a spring is proportional to the force exerted to stretch it (1660).
- Boyle's Law: For a fixed quantity of gas at constant temperature, pressure is inversely proportional to volume (1662).

- Newton's proposition on the prism: there is some kind of proportionality between refrangibility and colour of light (1672).
- Newton's Second Law of Motion: The acceleration of a body is proportional to the total force acting on it (1687).
- Newton's Law of Gravity: The force of gravity exerted by one body on another is proportional to the masses of each and inversely proportional to the square of the distance between them (1687).
- Newton's Law of Cooling: The rate of temperature loss from a body is proportional to the difference in temperature between the body and its surroundings (1701).

It was remarkable how tractable Nature proved to be not only to being mathematised, but mathematised in the simplest way possible, by laws of simple proportionality. That allows calculation of results with very simple mathematics, fortunately for both early scientists and for modern science students. What science teachers should emphasize is that the proportionalities are not in formulas imposed on reality, but in physical nature itself. The formulas merely describe a pre-existing reality.

Many other examples could be given of purely mathematical properties of things that play a crucial role in science. Symmetry has become crucial to science, especially physics, and the ability of symmetry arguments to generate contentful scientific truths is extraordinary. (Franklin 2017) Alternation, as in stripes and wallpaper patterns, is another recurrent mathematical theme in science; for example, in pendulum motion the gravitational cause is a matter of physics but oscillation itself is a mathematical property. Continuous variation (whether over time or in space), studied so successfully by the calculus, is a central theme in most of classical science, such as planetary motion and fluid flow. Randomness, in the sense of patternlessness, is central to any stochastic process from coin-throwing to traffic flow and stock markets.

What is essential for science teaching and for any appreciation of the big picture of how science works is some feel for the difference between a physical property and a mathematical one. That gives one some grasp of which methods will be needed to study the property: in particular, that for mathematical properties, mathematical methods such as conceptual analysis, definition and proof of theorems will play the main role.

## Science teaching: Awareness of mathematical necessities in reality

Science and science teaching rightly highlight the laws of nature, such as the law of gravity. It is generally understood that such laws have only an "empirical necessity", less than absolute. Indeed, the original point of calling them "laws", as became popular in the time of the early Royal Society, was to suggest they were commands laid down by God to which there could in principle be miraculous exceptions (Oakley 1961). But there are stronger, more absolute, necessities than that, also found directly in the real world. Scientists and science teachers need to be aware of the difference.

Consider what tiles I should order for my bathroom floor, which is close to a flat Euclidean plane. I can order square ones or hexagonal ones, as in the figures: the plane can be tiled in a regular fashion with those two shapes.

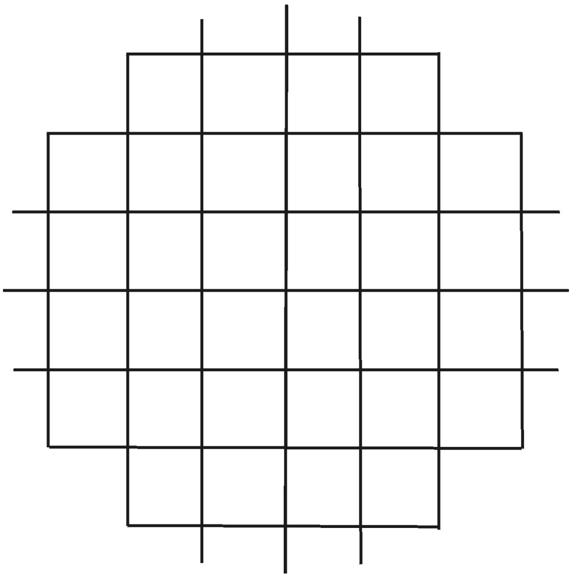

Fig 3. Tiling of the plane by identical squares

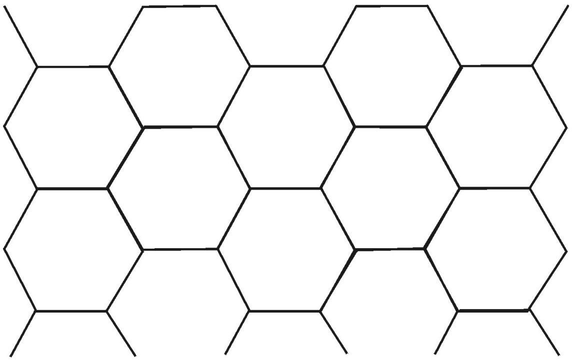

Fig 4. Tiling of the plane by identical regular hexagons

But there is no point ordering a load of regular pentagonal tiles. They just cannot be fitted to tile the bathroom floor. There are big pieces of floor left over between the tiles no matter how they are laid down, which defeats the purpose of tiling.

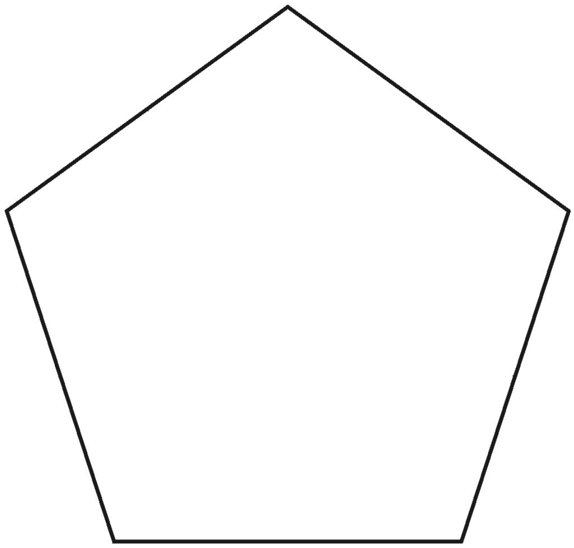

Fig 5. A regular pentagon, which cannot tile the plane

The necessity with which bathroom floors can be tiled with regular square or hexagonal tiles but not with pentagonal ones is stronger than that of the laws of nature. It is mathematical (Franklin 2014a, ch. 5).

Another example of an absolute, mathematical necessity (more exactly, impossibility) in the real world was discovered by Galileo during his efforts to establish the law of free fall. It is one of the most remarkable demonstrations of the power of a priori mathematical reasoning in dynamics. When first considering what law should be followed by falling heavy bodies (given they go faster as they fall), he wondered about how to distinguish between the two simplest theories: the perhaps most natural one that speed is proportional to distance travelled from the start, and the equally simple but perhaps less natural one that speed is proportional to time from the start (that is, the body is uniformly accelerated, which is the correct answer). Galileo realised, and was able to demonstrate, that the first theory needs no observations to refute it. It is absolutely impossible that acceleration should be proportional to the distance travelled (Norton and Roberts 2012).

From the falsity of the theory of the proportionality of speed to distance there does not follow, of course, the truth of the (true) alternative theory of the proportionality to time. There are other possible laws. But it leaves that theory as the natural simple alternative, thus guiding confirmation by experiment.

A third example has become well-known. It is a beautiful problem often used to introduce students to graph theory, Euler's eighteenth-century explanation of why it is impossible to walk over all of the seven bridges of Königsberg once and once only. The bridges connected two islands and two riverbanks as shown in the diagram. The citizens of Königsberg suspected from trial and error that it was not possible to walk over all the bridges, without walking over at least one of them twice.

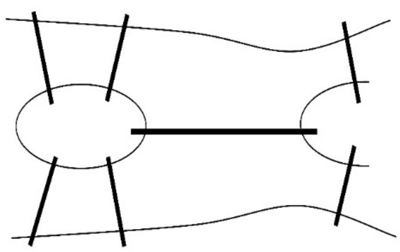

Fig 6. The seven bridges of Königsberg, connecting two banks and two islands

Euler proved they were right. His proof is purely in terms of a very general aspect of geometry – the topology or interconnections of the bridges and land areas. (Euler begins his paper by noting it belongs to a new, non-quantitative part of geometry, the "geometry of site"; the field is now called network topology.) There is no idealisation or approximation involved in drawing the diagram; although a simplified representation of the city, it contains all the relevant geometrical features and the proof applies directly to the system of real bridges and land areas, demonstrating an impossibility about physical reality (Franklin 2014a, ch. 13).

The necessities involved in tiling, in Galileo's discovery about speed and distance, and the Königsberg bridges, are mathematical ones and are not subject to miraculous exception. Science teaching is not complete unless such necessities are distinguished from the weaker "necessities" of natural laws, such as the law of gravity.

## Science teaching: Idealisation and approximation

Although we cannot go into it at length here, Aristotelian realism raises important questions about the nature of idealisation in science. In saying that mathematical laws of science are literally true of the observable world, modern Aristotelians to some extent make common cause with the Aristotelians of Galileo's day who objected that his ideal world of frictionless falling bodies was a fiction with no relevance to the real world. Modern Aristotelians do agree that idealisation is useful, but only on certain conditions. They can only work when the idealisation is an approximation to the real (usually more complicated) system, and the ideal model is "structurally stable", in the sense that results about it carry across, approximately, to the real situation that it approximates (Franklin, 2014a, 224-9).

Something of that idea was visible even in Galileo himself. In explaining Galileo's method of idealisation, Matthews (2006) records the objections of his Aristotelian patron, Guidobaldo del Monte: "When del Monte tells Galileo that he has done an experiment with balls in an iron hoop and the balls do not behave as Galileo asserts, Galileo replies that the hoop must not have been smooth enough, that the balls were not spherical enough and so on." That is only a sufficient reply if in fact making the hoop more smooth and the balls more spherical will cause the observed results to approximate more closely to those of the ideal model. Approximation is essential to idealisation.

## Conclusion

Aristotelian realism gives a new perspective (or rather, a very old perspective revived) on both mathematics itself and the role of mathematics in science. In holding that mathematics can apply directly to physical reality, it brings mathematics in closer contact with science. Its vision of mathematics as a contentful science of the world we live in, and one with necessary truths, allows mathematics to regain its rightful place at the centre of civilisation's achievements (Franklin, 2018). Its explanation of which properties exactly are mathematical (namely, quantitative and structural ones) allows philosophy of science and science teaching to properly understand the division of labour between mathematics and science, and hence the true role of mathematics in science.

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